J. Fluid Mech. (1969), vol. 38, part 2, pp. 365–374
Printed in Great Britain

Internal wave propagation in an inhomogeneous fluid of non-uniform depth

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(Received 20 January 1969)

An asymptotic solution is obtained to the problem of internal wave propagation in a horizontally stratified inhomogeneous fluid of non-uniform depth. It also applies to fluids which are not stratified, but in which the constant density surfaces have small slopes. The solution is valid when the wavelength is small compared to all horizontal scale lengths, such as the radius of curvature of a wavefront, the scale length of the bottom surface variations and the scale length of the horizontal density variations. The theory underlying the solution involves rays, a phase function satisfying the eiconal equation, and amplitude functions satisfying transport equations. All these equations are solved in terms of the rays and of the solution of the internal wave problem for a horizontally stratified fluid of constant depth. The theory is thus very similar to geometrical optics and its extensions. It can be used to treat problems of propagation, reflexion from vertical cliffs or from shorelines, refraction, determination of the frequencies and wave patterns of trapped waves, etc. For fluid of constant density, it reduces to the theory of Keller (1958). The theory is applied to waves in a fluid with an exponential density distribution on a uniformly sloping beach. The predicted wavelength is shown to agree well with the experimental result of Wunsch (1969). It is also applied to determine edge waves near a shoreline and trapped waves in a channel.

1. Introduction

The theory of internal gravity waves has been developed primarily for a horizontally stratified fluid of uniform depth, bounded above by a horizontal free surface and below by a rigid horizontal bottom. Fluids of non-uniform depth have been considered by Magaard (1962), Cox (1959), Cox & Sandstrom (1962) and Wunsch (1968, 1969). Wunsch considered waves in a fluid with an exponential density variation on a uniformly sloping bottom. We shall work out the theory for fluids for any stratification and any non-uniform depth. The resulting theory also applies to fluids with negligible velocity, in which the constant density surfaces have small slopes. It is even useful for horizontally stratified fluids of uniform depth, although then a simpler theory applies.

The theory we shall develop is an adaptation of that developed by Keller (1958) to treat surface wave propagation in a fluid of constant density and non-

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uniform depth. Therefore it applies when the fluid depth and the wave amplitude vary very little over a horizontal distance of one wavelength. Mathematically it is an asymptotic theory, valid when the wavelength is short compared to some horizontal scale length typical of the bottom surface, and also short compared to the radius of curvature of a wave front.

The theory involves rays, the eiconal equation for a phase function and transport equations for amplitude functions. Thus it is very similar to geometrical optics and its extensions, which yield an asymptotic theory of the wave equation. Consequently, it can be used for all the purposes for which that theory has been used. This includes the determination of the waves radiated from a source, the reflexion of waves from a vertical cliff or from a shoreline, the refraction of propagating waves, the diffraction of waves by islands, etc.

Shen, Meyer & Keller (1968) have recently determined the eigenfrequencies of trapped waves, such as edge waves, in fluid of constant density but non-uniform depth. They used the theory of Keller (1958) together with the method of Keller & Rubinow (1960) for finding eigenvalues of the reduced wave equation. We shall give a similar analysis using the present theory, and show that many of the results of Shen *et al.* (1968) can be carried over with slight modifications. We shall also apply our theory to the case treated by Wunsch (1968), and show that the predicted wavelength agrees with his experimental result (Wunsch 1969).

Like geometrical optics and other asymptotic theories of wave propagation the present theory fails at caustics, and it also fails at shorelines. It could be made valid at such places by the introduction of boundary layers, or by the uniform method of Kravtsov (1964) and Ludwig (1966). Boundary layers are also needed wherever some derivative of the depth is discontinuous. Without them, we can obtain only a finite number of terms in the expansion of the wave motion.

2. Formulation

Let $\rho_0(Y)$ be the density of a stably stratified incompressible inviscid fluid at rest,† bounded below by the rigid surface $Y = -H(X,Z)$ and above by the free surface $Y = 0$. Let $\mathbf{v} = (u,v,w)\,e^{-i\omega t}$ denote the velocity of a small amplitude time harmonic motion of the fluid, $\rho\,e^{-i\omega t}$ the corresponding change in density, $p\,e^{-i\omega t}$ the accompanying change in pressure, and $Y = \eta(X,Z)\,e^{-i\omega t}$ the resulting equation of the free surface. The linearized equations of motion, incompressibility and continuity satisfied by these quantities are:

$$i\omega\rho_0\mathbf{v} = \nabla p + (0, g\rho, 0). \tag{2.1}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{2.2}$$

$$-i\omega\rho + \mathbf{v}\cdot\nabla\rho_0 = 0. \tag{2.3}$$

† To apply the theory to be developed here to a slowly moving unstratified fluid, we assume that the slow motion is negligible. We also ignore the X, Z and t derivatives of the density $\rho_0(Y; X, Z, t)$. The dependence of $\rho_0(Y)$ on X, Z and t will not be shown explicitly.

The linearized kinematic and dynamic conditions on the free surface, and the condition of rigidity of the bottom are

$$-i\omega\eta(X,Z) = v(X,0,Z), \tag{2.4}$$

$$p(X,0,Z) = g\rho_0(0)\,\eta(X,Z), \tag{2.5}$$

$$\mathbf{v}[X,-H(X,Z),Z]\cdot\nabla[Y+H(X,Z)] = 0. \tag{2.6}$$

We seek solutions $\mathbf{v}$, ρ, p and η of (2.1)–(2.6).

From (2.1), (2.3) and (2.5) we obtain $\mathbf{v}$, ρ and η in terms of p, in the form

$$u = \frac{p_X}{i\omega\rho_0}, \quad v = -i\omega p_Y(g\rho_{0Y}+\omega^2\rho_0)^{-1}, \quad w = \frac{p_z}{i\omega\rho_0}, \tag{2.7}$$

$$\rho = \rho_{0Y}p_Y(\omega^2\rho_0+g\rho_{0Y})^{-1}, \tag{2.8}$$

$$\eta = \frac{p(X,0,Z)}{g\rho_0(0)}. \tag{2.9}$$

Now we use (2.7) and (2.9) to eliminate $\mathbf{v}$ and η from (2.2), (2.4) and (2.6), to obtain

$$\omega^2\rho_0[p_Y(g\rho_{0Y}+\omega^2\rho_0)^{-1}]_Y+\Delta p = 0, \tag{2.10}$$

$$p_Y(X,0,Z) = \left[\frac{\rho_{0Y}(0)}{\rho_0(0)}+\frac{\omega^2}{g}\right]p(X,0,Z), \tag{2.11}$$

$$\rho_0 p_Y + [(g/\omega^2)\rho_{0Y}+\rho_0]\,\nabla H\cdot\nabla p = 0 \quad \text{at} \quad Y = -H(X,Z). \tag{2.12}$$

In (2.10)–(2.12) and below, $\nabla = (\partial_X, \partial_Z)$ and $\Delta = \partial_X^2+\partial_Z^2$. Once we find p satisfying (2.10)–(2.12), $\mathbf{v}$, ρ and η can be found from (2.7) to (2.9).

It is convenient to introduce the dimensionless quantities x, y, z, h, α and β defined by

$$x = X/L, \quad y = \omega^2 Y/g, \quad z = Z/L, \quad h = \omega^2 H/g, \quad \alpha = 1+\frac{g\rho_{0Y}}{\omega^2\rho_0}, \quad \beta = \omega^2 L/g. \tag{2.13}$$

Here L denotes a typical horizontal scale length, and α is one minus the ratio of the square of the Väisälä frequency to ω^2. If we consider p, ρ_0, h and α to be functions of the new variables, then (2.10)–(2.12) become

$$\beta^2\left[p''-\frac{(\alpha\rho_0)'}{\alpha\rho_0}p'\right]+\alpha\Delta p = 0, \tag{2.14}$$

$$p'-\alpha p = 0 \quad \text{on} \quad y = 0, \tag{2.15}$$

$$\beta^2 p'+\alpha\nabla h\cdot\nabla p = 0 \quad \text{on} \quad y = -h(x,z). \tag{2.16}$$

Here and below the prime denotes differentiation with respect to y.

3. Asymptotic solution for β large

For β large we seek an asymptotic solution of (2.14)–(2.16), which is of the form

$$p(\mathbf{x},\beta) = A(\mathbf{x},\beta)\,\phi(y;x,z)\,e^{i\beta s(x,z)}. \tag{3.1}$$

The amplitude A and phase s are to be determined, while ϕ is defined to be a solution of the problem

$$\phi'' - \frac{(\alpha\rho_0)'}{\alpha\rho_0}\phi' - n^2(x,z)\alpha\phi = 0, \tag{3.2}$$

$$\phi' - \alpha\phi = 0 \quad \text{on} \quad y = 0, \tag{3.3}$$

$$\phi' = 0 \quad \text{on} \quad y = -h(x,z). \tag{3.4}$$

This is an eigenvalue problem for the ordinary differential equation (3.2) with eigenvalue $n^2(x,z)$, corresponding to wave propagation in a stratified fluid of uniform depth $h(x,z)$. Thus both n and ϕ depend on x and z, since h does and ρ_0 may. They can be found analytically if $\rho_0(y)$ is simple enough, and numerically otherwise. Each eigenfunction with $n^2 > 0$ represents a mode of propagation with the real propagation constant $\beta n(x,z)$. One of these modes is the surface wave and the others, if any, are internal waves. The eigenfunctions with $n^2 < 0$ represent evanescent or non-propagating modes. We shall assume that n^2 and ϕ are known, and try to determine A and s in terms of them.

Upon inserting (3.1) into (2.14)–(2.16) we obtain:

$$(i\beta)^2\left[\alpha(\nabla s)^2A\phi + \frac{(\alpha\rho_0)'}{\alpha\rho_0}(A\phi)' - (A\phi)''\right] + i\beta\alpha[2\nabla s\cdot\nabla(A\phi) + A\phi\Delta s] + \alpha\Delta(A\phi) = 0, \tag{3.5}$$

$$(A\phi)' - \alpha A\phi = 0 \quad \text{on} \quad y = 0, \tag{3.6}$$

$$\beta^2(A\phi)' + i\beta\alpha A\phi\nabla s\cdot\nabla h + \alpha\nabla h\cdot\nabla(A\phi) = 0 \quad \text{on} \quad y = -h. \tag{3.7}$$

By making use of (3.2)–(3.4) we can rewrite (3.5)–(3.7) as follows:

$$(i\beta)^2\left[\alpha A\phi\{(\nabla s)^2 - n^2\} + \frac{(\alpha\rho_0)'}{\alpha\rho_0}A'\phi - 2A'\phi' - A''\phi\right] + i\beta\alpha[2\nabla s\cdot\nabla(A\phi) + A\phi\Delta s] + \alpha\Delta(A\phi) = 0, \tag{3.8}$$

$$A' = 0 \quad \text{on} \quad y = 0, \tag{3.9}$$

$$(i\beta)^2A'\phi - i\beta\alpha A\phi\nabla s\cdot\nabla h - \alpha\nabla h\cdot\nabla(A\phi) = 0 \quad \text{on} \quad y = -h(x,z). \tag{3.10}$$

To solve (3.8)–(3.10) for A and s, we assume that A has the asymptotic expansion

$$A(\mathbf{x},\beta) \sim A_0(x,z) + \sum_{m=1}^{\infty}(i\beta)^{-m}A_m(\mathbf{x}). \tag{3.11}$$

We now insert (3.11) into (3.8)–(3.10) and then equate to zero the coefficient of each power of β in each equation. This yields the following system of equations for s and the A_m:

$$(\nabla s)^2 = n^2(x,z), \tag{3.12}$$

$$\phi A_m'' - \left[\phi\frac{(\alpha\rho_0)'}{\alpha\rho_0} - 2\phi'\right]A_m' = \alpha[2\nabla s\cdot\nabla(\phi A_{m-1}) + \phi A_{m-1}\Delta s] + \alpha\Delta(\phi A_{m-2}), \tag{3.13}$$

$$A_m' = 0 \quad \text{on} \quad y = 0, \tag{3.14}$$

$$\phi A_m' = \alpha\phi A_{m-1}\nabla s\cdot\nabla h + \alpha\nabla h\cdot\nabla(\phi A_{m-2}) \quad \text{on} \quad y = -h(x,z). \tag{3.15}$$

Equations (3.13)–(3.15) hold for $m \geqslant 1$ and in them $A_m \equiv 0$ if $m < 0$.

Equation (3.12) is the well-known eiconal equation of geometrical optics, in which $n^2(x,z)$ plays the role of the refractive index. Therefore it can be solved by means of rays. These are curves normal to the wave fronts s = constant, and they are the characteristics of (3.12). Let the parametric equation of a ray be $x(\sigma)$, $z(\sigma)$, in which the parameter σ denotes arc-length along the ray increasing in the direction of propagation. Then the rays are the solutions of the ray equations

$$x_{\sigma\sigma} = nn_x, \quad z_{\sigma\sigma} = nn_z, \quad x_\sigma^2 + z_\sigma^2 = 1. \tag{3.16}$$

Along a ray s is given by

$$s[x(\sigma),z(\sigma)] = s[x(\sigma_0),z(\sigma_0)] + \int_{\sigma_0}^{\sigma} n[x(\sigma'),z(\sigma')]\,d\sigma'. \tag{3.17}$$

Once s has been found, (3.13)–(3.15) constitute a recursive system of equations for the successive determination of the A_m, starting with $m = 1$. For $m = 0$, (3.13)–(3.15) are satisfied trivially, because $A_0' = A_0'' = 0$, since A_0 is independent of y by assumption, and $A_{-1} = A_{-2} = 0$ by definition. To solve these equations for $m \geqslant 1$, we multiply (3.13) by $\phi/\alpha\rho_0$, and integrate the result from $y = 0$ to obtain

$$\frac{\phi^2 A_m'(y)}{\alpha\rho_0} = -\int_y^0 \frac{\phi}{\rho_0}[2\nabla s\cdot\nabla(\phi A_{m-1}) + \phi A_{m-1}\Delta s + \Delta(\phi A_{m-2})]\,dy. \tag{3.18}$$

In (3.18) and below, we indicate only the argument y of $A_m(x,y,z)$, and suppress x and z. To obtain (3.18) we have put $A_m'(0) = 0$, in view of (3.14). We now multiply (3.18) by $\alpha\rho_0/\phi^2$ and integrate the result with respect to y from $-h$ to y, obtaining:

$$A_m(y) = A_m(-h) - \int_{-h}^{y} \frac{\alpha\rho_0}{\phi^2}\int_{y'}^{0}\frac{\phi}{\rho_0}[2\nabla s\cdot\nabla(\phi A_{m-1}) + \phi A_{m-1}\Delta s + \Delta(\phi A_{m-2})]\,dy''\,dy'. \tag{3.19}$$

To obtain $A_m(-h)$, which occurs in (3.19) ,we set $y = -h$ in (3.18), use (3.15) and replace m by $m+1$, to find

$$\frac{\phi^2(-h)}{\rho_0(-h)}A_m(-h)\nabla s\cdot\nabla h + \frac{\phi(-h)}{\rho_0(-h)}\nabla h\cdot\nabla[\phi(-h)A_{m-1}(-h)]$$
$$= -\int_{-h}^{0}\frac{\phi}{\rho_0}[2\nabla s\cdot\nabla(\phi A_m) + \phi A_m\Delta s + \Delta(\phi A_{m-1})]\,dy. \tag{3.20}$$

This is not a useful expression for $A_m(-h)$, because $A_m(y)$ also occurs in the integrand. Therefore we substitute (3.19) for $A_m(y)$ in the integrand, and obtain

$$\frac{\phi^2(-h)}{\rho_0(-h)}A_m(-h)\nabla s\cdot\nabla h + \int_{-h}^{0}\frac{\phi}{\rho_0}\{2\nabla s\cdot\nabla[\phi A_m(-h)] + \phi A_m(-h)\Delta s\}\,dy = B_{m-1}. \tag{3.21}$$

Here $B_{m-1}(x,z)$ is defined by

$$B_{m-1}(x,z) = -\frac{\phi(-h)}{\rho_0(-h)}\nabla h\cdot\nabla[\phi(-h)A_{m-1}(-h)]$$
$$-\int_{-h}^{0}\frac{\phi}{\rho_0}[2\nabla s\cdot\nabla(\phi I_{m-1}) + \phi I_{m-1}\Delta s + \Delta(\phi A_{m-1})]\,dy. \tag{3.22}$$

In (3.22) $I_{m-1}(\mathbf{x}) = A_m(y) - A_m(-h)$ is given by the integral on the right side of (3.19), in terms of A_{m-1} and A_{m-2}. Thus, $B_{m-1}(x,z)$ is expressed in terms of A_{m-1} and A_{m-2}, so it can be considered to be known. Then (3.21) can be viewed as an equation for $A_m(-h)$.

We can rewrite (3.21) in the form

$$2\psi\nabla s\cdot\nabla A_m(-h) + A_m(-h)\nabla s\cdot\nabla\psi + \psi A_m(-h)\Delta s = B_{m-1}. \tag{3.23}$$

Here ψ is defined by

$$\psi(x,y) = \int_{-h}^{0}\frac{\phi^2}{\rho_0}\,dy. \tag{3.24}$$

Now (3.23) can be written as an ordinary differential equation along a ray, since the directional derivative $\nabla s\cdot\nabla$ is equal to $n\,d/d\sigma$. This is so because the magnitude of ∇s is n, as we see from (3.12), and the ray direction is normal to the curves $s=$ constant, as it is parallel to ∇s. Therefore we can write (3.23) as

$$2n\frac{d}{d\sigma}[A_m(-h)\psi^{\frac{1}{2}}] + [A_m(-h)\psi^{\frac{1}{2}}]\Delta s = \psi^{-\frac{1}{2}}B_{m-1}. \tag{3.25}$$

The solution $A_m(-h,\sigma)$ of (3.25), at the point $x(\sigma)$, $z(\sigma)$ on a ray, is

$$A_m(-h,\sigma) = A_m(-h_0,\sigma_0)\frac{\psi^{\frac{1}{2}}(\sigma_0)}{\psi^{\frac{1}{2}}(\sigma)}\exp\left(-\frac{1}{2}\int_{\sigma_0}^{\sigma}\frac{\Delta s}{n}\,d\sigma'\right) + \tfrac{1}{2}\psi^{-\frac{1}{2}}(\sigma)\int_{\sigma_0}^{\sigma}\exp\left(-\frac{1}{2}\int_{\sigma''}^{\sigma}\frac{\Delta s}{n}\,d\sigma'\right)\frac{\psi^{-\frac{1}{2}}(\sigma'')}{n(\sigma'')}B_{m-1}\,d\sigma''. \tag{3.26}$$

The exponential factor in (3.26) can be evaluated in terms of geometrical quantities with the result (Luneburg 1944):

$$\exp\left(-\frac{1}{2}\int_{\sigma_0}^{\sigma}\frac{\Delta s}{n}\,d\sigma'\right) = \left[\frac{n(\sigma_0)}{n(\sigma)}\frac{da(\sigma_0)}{da(\sigma)}\right]^{\frac{1}{2}}. \tag{3.27}$$

Here $da(\sigma)$ denotes the width of a narrow strip of rays at the point $x(\sigma)$, $y(\sigma)$ on a ray, and $da(\sigma_0)$ denotes the width of the same strip at $x(\sigma_0)$, $z(\sigma_0)$ on the same ray. The ratio $da(\sigma)/da(\sigma_0)$ is also the Jacobian of the mapping by means of rays of the wavefront through $x(\sigma_0)$, $z(\sigma_0)$ on the wavefront through $x(\sigma)$, $z(\sigma)$. Upon using (3.27) in (3.26) we can write $A_m(-h,\sigma)$ in the form

$$A_m(-h,\sigma) = A_m(-h,\sigma_0)\left[\frac{\psi(\sigma_0)\,n(\sigma_0)\,da(\sigma_0)}{\psi(\sigma)\,n(\sigma)\,da(\sigma)}\right]^{\frac{1}{2}} + \tfrac{1}{2}[n(\sigma)\psi(\sigma)]^{-\frac{1}{2}}\int_{\sigma_0}^{\sigma}[n(\sigma'')\psi(\sigma'')]^{-\frac{1}{2}}\left[\frac{da(\sigma'')}{da(\sigma)}\right]^{\frac{1}{2}}B_{m-1}\,d\sigma''. \tag{3.28}$$

Then $A_m(y,\sigma)$ is given by (3.19), in which $A_m(-h,\sigma)$ is given by (3.28). By using the result (3.19) for $A_m(y,\sigma)$ in (3.11), we obtain $A(\mathbf{x},\beta)$. Then $p(\mathbf{x},\beta)$ is given by (3.1), in which A is given by (3.11), $s(x,y)$ by (3.17); and $\phi(y;x,z)$ is a solution of (3.2)–(3.4).

The leading term in $A(\mathbf{x},\beta)$ is $A_0(x,z)$, which is independent of y. Therefore A_0 is given by (3.28) with $m=0$. Since $A_m = 0$ if $m<0$, it follows from the definition (3.22) that $B_{-1}=0$. Then (3.28) yields

$$A_0(\sigma) = A_0(\sigma_0)\left[\frac{\psi(\sigma_0)\,n(\sigma_0)\,da(\sigma_0)}{\psi(\sigma)\,n(\sigma)\,da(\sigma)}\right]^{\frac{1}{2}}. \tag{3.29}$$

To see the physical meaning of (3.29) we rewrite it in the form

$$A_0^2(\sigma)\,\psi(\sigma)\,n(\sigma)\,da(\sigma) = A_0^2(\sigma_0)\,\psi(\sigma_0)\,n(\sigma_0)\,da(\sigma_0). \tag{3.30}$$

This is a conservation equation, which states that the energy flux, proportional to $A^2\psi n\,da$, is constant along an infinitesimal strip of rays.

The leading term in the expansion of p is obtained by using (3.29) for A_0 and (3.17) for s in (3.1). The result is

$$p[x(\sigma),y,z(\sigma),\beta] \sim A_0[x(\sigma_0),z(\sigma_0)]\left[\frac{\psi(\sigma_0)\,n(\sigma_0)\,da(\sigma_0)}{\psi(\phi)\,n(\sigma)\,da(\sigma)}\right]^{\frac{1}{2}}\phi(y;x,z)$$
$$\times\exp\left[i\beta\left\{s[x(\sigma_0),z(\sigma_0)]+\int_{\sigma_0}^{\sigma} n[x(\sigma'),z(\sigma')]\,d\sigma'\right\}\right]. \tag{3.31}$$

The amplitude $A_0[x(\sigma_0),z(\sigma_0)]$, and phase $s[x(\sigma_0),z(\sigma_0)]$ at the point $x(\sigma_0),z(\sigma_0)$ on a ray, are arbitrary. From (3.31) and (2.7)–(2.9), we can obtain the leading terms in $\mathbf{v}$, ρ and η. The leading terms in u and w are obtained by differentiating the exponent in (3.31), so that (2.7) yields

$$(u,w) = \frac{1}{i\omega\rho_0}(p_X,p_Z) \sim \frac{\beta}{\omega\rho_0 L}p\nabla s. \tag{3.32}$$

This shows that the horizontal velocity is in the ray direction ∇s at any depth y. For v and ρ, (2.7), (2.8) and (3.31) yield

$$v = \frac{-i\omega\beta}{(\omega^2\rho_0+g\rho_{0Y})L}p_y \sim \frac{-i\beta}{\omega\rho_0\alpha L}\frac{\phi_y p}{\phi}, \tag{3.33}$$

$$\rho = \frac{-\rho_{0y}\beta^2}{\omega^2\rho_0\alpha L^2}p_y \sim \frac{-\beta\rho_{0y}}{g\rho_0\alpha L}\frac{\phi_y p}{\phi}. \tag{3.34}$$

The height η of the free surface is, from (2.9),

$$\eta = \frac{p(x,0,z)}{g\rho_0(0)}. \tag{3.35}$$

In (3.32)–(3.35), p is given by (3.31).

In the special case $\rho_0(y)$ = constant, all the results obtained above reduce to those obtained by Keller (1958) for the uniform density case.

4. An example

Let us apply our theory to a fluid with the exponential density distribution

$$\rho_0(Y) = \rho_{00}\exp(-N^2Y/g). \tag{4.1}$$

Here N is the constant Väisälä frequency. Then (2.13) yields for α the constant value

$$\alpha = 1-N^2/\omega^2. \tag{4.2}$$

Now (3.2) becomes, upon recalling from (2.13) that $Y = yg/\omega^2$,

$$\phi'' + (N^2/\omega^2)\,\phi' - n^2\alpha\phi = 0. \tag{4.3}$$

A solution of (4.3) satisfying (3.4) (i.e. $\phi' = 0$ at $y = -h$) is

$$\phi(y) = \frac{1}{\lambda_1}\exp[\lambda_1(y+h)] - \frac{1}{\lambda_2}\exp[\lambda_2(y+h)] \tag{4.4}$$

$$\lambda_{\frac{1}{2}} = -\frac{N^2}{2\omega^2} \pm i\left(-\alpha n^2 - \frac{N^4}{4\omega^4}\right)^{\frac{1}{2}}. \tag{4.5}$$

At the top surface $y = 0$, we shall suppose that there is a rigid lid, so that (3.3) is replaced by

$$\phi' = 0 \quad \text{on} \quad y = 0. \tag{3.3$'$}$$

When (4.4) is used in (3.3′), it yields the dispersion equation,

$$\left(-\alpha n^2 - \frac{N^4}{4\omega^4}\right)^{\frac{1}{2}} h = m\pi \quad (m = 1, 2, \ldots). \tag{4.6}$$

Solving (4.6) for n, and using (4.2) for α, yields

$$n = \left(\frac{N^2}{\omega^2} - 1\right)^{-\frac{1}{2}}\left[\left(\frac{m\pi}{h}\right)^2 + \frac{N^4}{4\omega^4}\right]^{\frac{1}{2}}. \tag{4.7}$$

When (4.7) is used in (4.5), it yields

$$\lambda_{\frac{1}{2}} = -\frac{N^2}{2\omega^2} \pm \frac{m\pi i}{h}. \tag{4.8}$$

The wave-number in the original variables is $\beta n/L$, and from (4.7) this is

$$\frac{\beta n}{L} = \left(\frac{N^2}{\omega^2} - 1\right)^{-\frac{1}{2}}\left[\left(\frac{m\pi}{H}\right)^2 + \frac{N^4}{4g^2}\right]^{\frac{1}{2}}. \tag{4.9}$$

It is of interest to compare (4.9) with the measured values of the horizontal wavelength $\lambda = 2\pi L/\beta n$ for waves in an exponentially stratified liquid over a uniformly sloping bottom with a rigid lid (Wunsch 1969). From (4.9) we have

$$\lambda = \frac{2H}{m}\left(\frac{N^2}{\omega^2} - 1\right)^{\frac{1}{2}}\left[1 + \left(\frac{N^2 H}{2\pi m g}\right)^2\right]^{-\frac{1}{2}}. \tag{4.10}$$

Except for the last factor, which is practically unity in the experimental case, this result is the same as that of Wunsch. In the experiment, $H = 0{\cdot}112x$, with x denoting distance from the shoreline, $(N^2/\omega^2 - 1)^{-\frac{1}{2}} = 1{\cdot}68$, the Väisälä period was approximately 3 sec, so that $N = \frac{2}{3}\pi$ sec^{-1} and $g = 980$ cm sec^{-2}. Thus, with $m = 1$, (4.10) yields

$$\lambda = 0{\cdot}133x\,[1 + O(10^{-8}x^2)]^{-\frac{1}{2}} \approx 0{\cdot}133x \quad (x \ll 10^4). \tag{4.11}$$

This result (4.11) agrees fairly well with the experimental results, as is shown in figure 1.

From (3.24), (4.4) and (4.8) we have

$$\psi = \int_{-h}^{0} \frac{\phi^2}{\rho_0}\,dy = \frac{-2h^3}{(m\pi)^2\rho_{00}}\exp(-N^2h/\omega^2)\left[1 + \left(\frac{N^2 h}{2m\pi\omega^2}\right)^2\right]^{-1}. \tag{4.12}$$

Now we can evaluate the leading term of p, given by (3.31), by using (4.12) for ψ, (4.7) for n and (4.4) for ϕ. If we denote quantities evaluated at σ_0 by affixing a subscript zero, the result can be written as

$$p \sim A_0 \frac{h_0}{h} \exp\{-N^2 h_0/2\omega^2\} \left[1+\left(\frac{N^2 h}{2m\pi\omega^2}\right)^2 \bigg/ 1+\left(\frac{N^2 h_0}{2m\pi\omega^2}\right)^2\right]^{\frac{1}{4}} \left(\frac{da_0}{da}\right)^{\frac{1}{2}} (-1)^m$$
$$\times \exp\left\{-\frac{N^2}{2\omega^2}y\right\} \left(\frac{\exp\{im\pi y/h\}}{\lambda_1} - \frac{\exp\{-im\pi y/h\}}{\lambda_2}\right) \exp\left\{i\beta\left(s_0 + \int_{\sigma_0}^{\sigma} n\, d\sigma\right)\right\}. \tag{4.13}$$

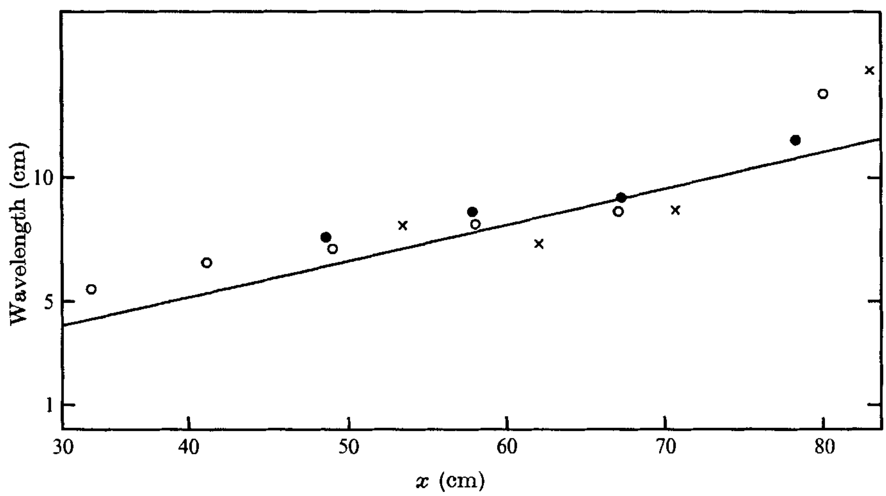

FIGURE 1. The wavelength at the distance x from the shoreline for waves in a fluid with an exponential density distribution over a uniformly sloping bottom. The straight line, based upon the present theory, is given by (4.11). The points are the measured values of Wunsch (1969). In the experiment $(N^2/\omega^2 - 1)^{-\frac{1}{2}} = 1{\cdot}68$ and $H/x = 0{\cdot}112$.

If $N^2 h/\omega^2 \ll 1$, (4.13) can be simplified to

$$p \sim \frac{2(-1)^m}{m\pi i} A_0 h_0 \left(\frac{da_0}{da}\right)^{\frac{1}{2}} \cos\left(\frac{m\pi y}{h}\right) \exp\left\{i\beta\left[s_0 + m\pi\left(\frac{N^2}{\omega^2}-1\right)^{-\frac{1}{2}} \int_{\sigma_0}^{\sigma} \frac{d\sigma}{h}\right]\right\}. \tag{4.14}$$

For two-dimensional cases $da_0/da = 1$. In two dimensions with a uniformly sloping bottom we can set $\sigma = x$ and $h = \gamma x$. We can also set $p_0 = \exp\{i\beta s_0\}\, 2(-1)^m A_0 h_0/m\pi i$. Then (4.14) becomes

$$p \sim p_0 \cos\left(\frac{m\pi y}{\gamma x}\right) \left(\frac{x}{x_0}\right)^{i\beta m\pi/\delta(N^2/\omega^2-1)^{-\frac{1}{2}}}. \tag{4.15}$$

This represents a wave travelling in the direction of increasing x, and its complex conjugate $\bar{p}$ represents a wave travelling in the opposite direction.

5. Trapped waves

As another application of this theory, we shall determine the frequencies of trapped waves in a stratified fluid of non-uniform depth. Let us consider first the case in which the fluid lies in the channel $x > 0$, $0 \leqslant z \leqslant b$, with a shoreline at $x = 0$. We assume that the depth is independent of z, so $h = h(x)$. Therefore the eigenvalue $n = n(x)$ is also independent of z; then trapped modes can occur. Each such mode is oscillatory between the shoreline $x = 0$ and a caustic at $x = a$, and decays exponentially for $x > a$.

The eigenfrequencies of the modes can be found by adapting equations (31) and (32) of Shen *et al.* (1968). To do so, we replace M by $\beta^{\frac{1}{2}}$, $k(x)$ by $n(x)$, and set $c = n(a)$. Then those equations become the following equations for β and a:

$$2\beta^{\frac{1}{2}}\int_0^a [n^2(x)-n^2(a)]^{\frac{1}{2}}\,dx = 2\pi(n_1+\tfrac{1}{2}) \quad (n_1 = 0,1,2,\ldots), \tag{5.1}$$

$$2\beta^{\frac{1}{2}}n(a)b = 2\pi n_2 \quad (n_2 = 1,2,\ldots). \tag{5.2}$$

The solutions of these equations yield the caustic distance aL and the eigenfrequency $\omega = (g\beta/L)^{\frac{1}{2}}$ for each mode, where L is the unit of length introduced in (2.13). The modes are labelled by the two integers n_1 and n_2. When the channel walls at $z = 0$ and $z = b$ are absent, then (5.1) alone determines the relation between β and a for edge waves along a shoreline. The frequency spectra determined by (5.1) and (5.2) are discussed in detail for fluid of uniform density by Shen *et al.* (1968); much of that discussion is applicable to the present case. In the same way that we adapted (31) and (32) to obtain (5.1) and (5.2), we can adapt many of the other results of Shen *et al.* to obtain trapped waves in circular basins, around circular islands, above submerged peaks, etc.

This research was supported by the Office of Naval Research, Contract No. N00014–67–A–0467–006.

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